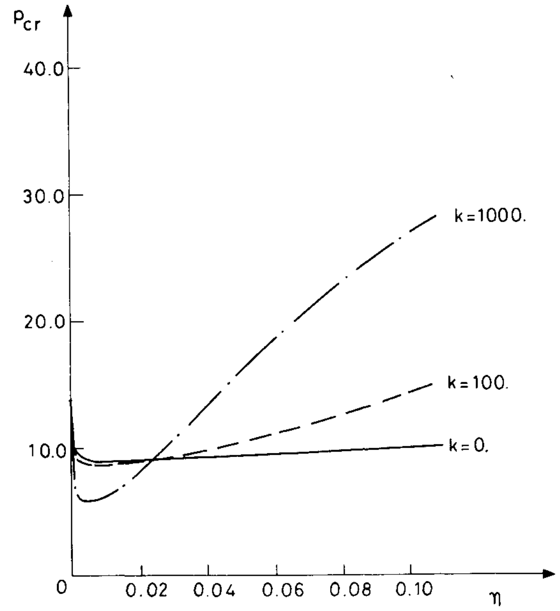

Fig. 4 Influence of the viscoelastic damping coefficient on the critical load for several values of the elastic foundation modulus ($I_e = 0.0044$).

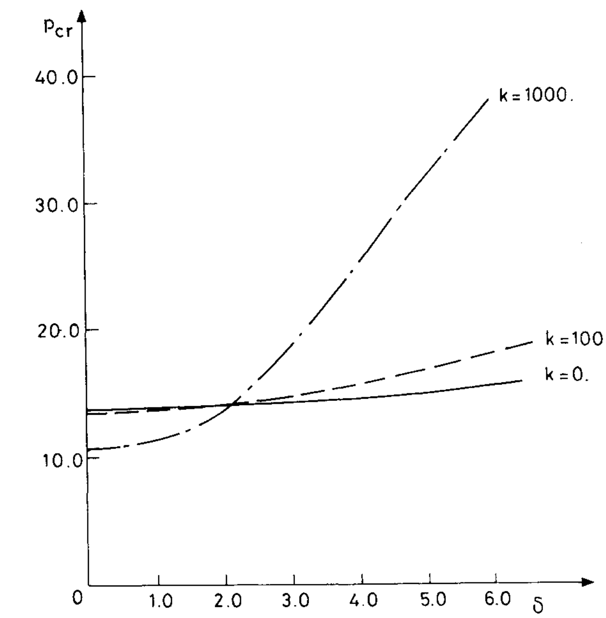

Fig. 5 Influence of the viscous damping coefficient on the critical load for several values of the elastic foundation modulus ($I_e = 0.0044$).

($\omega = 11.0$), which was first computed by Beck.[4] Starting from the known Beck's solution, one can increase the inertia of the beam (I_e) slowly to get a converged solution for the preceding problems. Figure 1 displays the variation of the critical load and frequency at beam flutter as the inertia of the beam increases. It is seen that the effect of shear deformation and rotatory inertia is to reduce the critical load (destabilizing effect) and frequency at which instability will occur. The critical loads obtained by the present method are in excellent agreement with the results reported in Ref. 1. The frequencies obtained by the finite element method of Ref. 1 are slightly higher than the frequencies computed by the present method (see Fig. 1). Figure 2 shows the variation of the critical load p as a function of increasing the elastic foundation modulus k. The present results confirm the phenomenon reported in Ref. 1 that the elastic foundation reduces the critical load at which the column will undergo dynamic instability.

The effect of viscoelastic damping η coupled with the elastic foundation modulus is plotted in Figs. 3 and 4 for $I_e = 0.0001$ and 0.0044, respectively. The striking phenomenon presented in Figs. 3 and 4 is the sharp reduction in the critical load p for a very lightly damped beam. A similar phenomenon is reported in Ref. 3 for the B-E beam. For shorter beams (higher values of I_e), the reduction in the critical load for very lightly damped beams is more pronounced for higher values of elastic foundation k (see Fig. 4). The results of Fig. 4 show that for lightly damped beams ($\eta < 0.025$) the effect of the elastic foundation is destabilizing, while for higher damping ($\eta > 0.025$) the elastic foundation has a stabilizing effect.

The viscous damping has a stabilizing effect for the B-E beam (small values of I_e), as reported in Ref. 3. Figure 5 shows the influence of the viscous damping coefficient on the critical load for a shorter beam ($I_e = 0.0044$) for several values of the elastic foundation modulus. The results of Figs. 4 and 5 show that for shorter beams and light damping (viscous and viscoelastic), the elastic foundation has a destabilizing effect. Increasing the damping coefficient beyond a certain value causes the elastic foundation to stabilize the system (a higher critical load for a higher elastic foundation coefficient).

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Influence of Mass Representation on the Equations of Motion for Rotating Structures

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Introduction

CONVENTIONAL analysis techniques are not applicable in the case of an elastic structure experiencing significant angular motion. This is of interest because numerous structural configurations such as spinning satellites, rotating shafts, and rotating linkages fall into this category. The analysis of these rotating structures differs from that of stationary structures due to the complexity of the accelerations

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acting throughout the system. In addition to the accelerations resulting from elastic structural deformations, contributions due to Coriolis and centripetal acceleration may be of significance. Also, the stiffness characteristics of the structure may be modified by the internal loads induced by the centrifugal forces.

Background

The classical stiffening effect experienced by a rotating structure is an often discussed topic in many vibration textbooks. As an example, in Refs. 1 and 2 the centrifugal force effects are retained in the modal analysis of a rotating radial beam. Presented in Ref. 3 is an extensive review of the structural dynamics analysis techniques for rotating turbomachinery blades. The majority of the reviewed work was concerned with radial-type structures whose displacements normal to the axis of rotation are small. Even in this situation, the influence of the complicated accelerations acting on the rotating structure has been recognized as is discussed in Refs. 4 and 5.

However, there are structural configurations for which the assumption of small displacements normal to the axis of rotation is not valid. An example of this situation is the dynamic analysis of a vertical axis wind turbine discussed in Ref. 6. Presented in this paper is the direct formulation of Coriolis and centripetal acceleration "finite elements" that are incorporated in the analysis flow. Flexible appendages on rotating spacecraft are a second example where motions normal to the axis of rotation are not necessarily small. These space structures can be quite large and relatively flexible, offset from the axis of rotation, and at an arbitrary orientation with respect to the rotational axis.

Early interest in this problem for space vehicles is discussed in Ref. 7. In Ref. 7, an extensive description is presented of the modal analysis problem for an elastic structure attached to a rotating base. Included are comments on the relative merits of various mathematical models for the elastic structure and the modal analysis results for a discrete spring-mass system having specific orientations with respect to the axis of rotation. The influence of rotation on the natural frequencies, mode shapes, and stability of a beam are discussed in Ref. 8. The effects of beam orientation and location with respect to the axis of rotation are presented in this paper. A diagonal mass representation was assumed for the analysis of Ref. 8.

Mass Representation

The general structural configuration of interest is shown in Fig. 1. Here we have a flexible structure attached to a rigid base that is rotating at constant angular velocity about an axis fixed in inertial space. As discussed in Ref. 9, an added complication arises in the system equations of motion when the

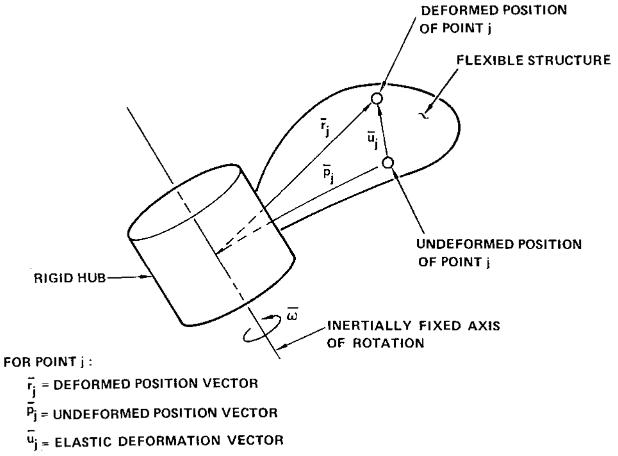

Fig. 1 Rotating structure.

mass representation of the rotating structure is nondiagonal. This point will be expanded upon in the following section.

The existence of such mass representations is not unusual since standard structural analysis techniques often result in nondiagonal mass matrices. For example, the mass matrix associated with a consistent mass formulation[10] is nondiagonal. In addition, when the Guyan reduction procedure[11] is applied, the resulting mass matrix is often nondiagonal. The Guyan reduction technique involves condensing the potentially hundreds of degrees of freedom in a finite element structural idealization to a limited number of active degrees of freedom for dynamic analysis. The condensed mass matrix will likely be nondiagonal even if the original matrix was diagonal.

As will be discussed in the following section, the use of the diagonal mass representation offers the advantage of simplifying the equations of motion for a rotating structure. However, for the nonrotating case, improved accuracy for a given finite element model size is obtained with a consistent mass matrix.[12] These differences in the nonrotating analysis results prompted this evaluation as to the influence of the mass representation on the form of the equations of motion for a rotating structure.

Equations of Motion Development

In matrix notation, the system kinetic energy is given as

$$T = \tfrac{1}{2}\dot{R}^T M \dot{R} \tag{1}$$

where M is the system mass matrix and $\dot R$ a column matrix defining the inertial velocities of the finite element node points located throughout the structure. Referring to Fig. 1, we see that the matrix $\dot R$ is of the form

$$\dot{R} = \begin{Bmatrix} \dot r_1 \\ \dot r_2 \\ \vdots \\ \dot r_j \end{Bmatrix} \tag{2}$$

Here each $\dot r_j$ submatrix contains the inertial velocity vector at the jth node point. The mass properties M of the structure are related to the three translational degrees of freedom at each node point.

The displacement vector $\bar r_j$ is made up of the undeformed position vector $\bar p_j$ and the elastic deformation vector $\bar u_j$

$$\bar r_j = \bar p_j + \bar u_j \tag{3}$$

The inertial velocity of point j thus becomes

$$\dot{\bar r}_j = \mathring{\bar p}_j + \mathring{\bar u}_j + \bar\omega \times (\bar p_j + \bar u_j) \tag{4}$$

The closed dot represents time differentiation with respect tc the inertial reference frame and the open dot the time differentiation with respect to a coordinate system fixed in the rotating structure. Note that $\bar p_j$ is constant and thus Eq. (4) becomes

$$\dot{\bar r} = \mathring{\bar u}_j + \bar\omega \times (\bar p_j + \bar u_j) \tag{5}$$

In matrix notation, Eq. (5) may be written as

$$r_j = \mathring{u}_j + \tilde\omega (p_j + u_j) \tag{6}$$

The expression $\tilde\omega(p_j + u_j)$ is the matrix equivalent to the vector cross product of Eq. (5) where $\tilde\omega$ is a skew symmetric matrix of the form

$$\tilde\omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{7}$$

Returning to the matrix expression [Eq. (2)] for all the inertial velocities throughout the structure, in combination with Eq. (6), we have

$$\dot{R} = \mathring{U} + \tilde{\Omega}(P+U) \tag{8}$$

where P and U are column matrices representing the undeformed locations and elastic deformations of all the structural node points. The matrix $\tilde{\Omega}$ is the vector cross-product operator expressed as

$$\tilde{\Omega} = \begin{bmatrix} [\tilde{\omega}] & & 0 \\ & [\tilde{\omega}] & \\ 0 & & [\tilde{\omega}] \end{bmatrix} \tag{9}$$

Combining Eqs. (8) and (1) results in the following expression for the kinetic energy:

$$\begin{aligned} T = {} & \tfrac{1}{2}\mathring{U}^T M \mathring{U} + \tfrac{1}{2}\omega^T \tilde{P}^T M \tilde{P}\omega \\ & + \tfrac{1}{2}\omega^T \tilde{U}^T M \tilde{U}\omega + \tfrac{1}{2}\omega^T [\tilde{P}^T M \tilde{U} + \tilde{U}^T M \tilde{P}]\omega \\ & - \omega^T (\tilde{P}^T + \tilde{U}^T) M \mathring{U} - \mathring{U}^T M (\tilde{P} + \tilde{U})\omega \end{aligned} \tag{10}$$

The system potential energy is expressed as

$$V = \tfrac{1}{2} U^T K U \tag{11}$$

where K is the system stiffness matrix.

Combining Eqs. (10) and (11) with the Lagrangian equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \mathring{U}} - \frac{\partial T}{\partial U} + \frac{\partial V}{\partial U} = 0 \tag{12}$$

and performing the indicated operations yields the equations of motion for the elastic displacements. These system equations of motion become

$$M\overset{\circ\circ}{U} + (M\tilde{\Omega} + \tilde{\Omega}M)\mathring{U} + (K + \tilde{\Omega}M\tilde{\Omega})U = -\tilde{\Omega}M\tilde{\Omega}P \tag{13}$$

with no externally applied loads.

Equation (13) represents the general elastic deflection equations of motion for a flexible structure rotating at a constant rate. The equations apply for either a diagonal or nondiagonal mass matrix. In these equations, $(M\tilde{\Omega} + \tilde{\Omega}M)U$ is due to Coriolis accelerations, $\tilde{\Omega}M\tilde{\Omega}\mathring{U}$ is due to centripetal acceleration, and $-\tilde{\Omega}M\tilde{\Omega}P$ is the centrifugal force due to the constant spin rate. The steady-state deflected or equilibrium position of the elastic structure due to the presence of Ω may be obtained from

$$(K + \tilde{\Omega}M\tilde{\Omega})U_s = -\tilde{\Omega}M\tilde{\Omega}P \tag{14}$$

Internal loads will be built up in the structure in going from its undeformed position to the new equilibrium configuration as defined by Eq. (14). These steady-state internal loads or centrifugal force effects modify the stiffness characteristics of the structure through the introduction of geometric stiffness terms. Several references, such as Przemieniecki,[13] present techniques for evaluation of this geometric stiffness matrix represented symbolically here as K_G.

Incorporating the concepts of geometric stiffness, Eq. (13) becomes

$$M\overset{\circ\circ}{U} + (M\tilde{\Omega} + \tilde{U}M)\mathring{U} + (K + K_G + \tilde{\Omega}M\tilde{\Omega})U = 0 \tag{15}$$

It must be remembered that the U in Eq. (15) now represent elastic deflections measured from the steady-state position defined by Eq. (14). If the mass matrix is diagonal and the three lumped or point masses for each node point have the same value, we have

$$M\tilde{\Omega} = \tilde{\Omega}M \tag{16}$$

In this case, Eq. (15) takes the form

$$M\overset{\circ\circ}{U} + 2\tilde{\Omega}\tilde{\Omega}M\mathring{U} + (K + K_G + \tilde{\Omega}\tilde{\Omega}M)U = 0 \tag{17}$$

These relationships of Eqs. (15) and (17) define the free vibration or eigenvalue problem that must be addressed to evaluate the influence of rotation on the modal characteritics on a rotating elastic structure.

Modal Analysis

For modal analysis of the rotating structure, the eigenvalue problem for Eq. (15) may be written as

$$\dot{Q} + DQ = 0 \tag{18}$$

where

$$Q = \left\{ \frac{\mathring{U}}{U} \right\} \tag{19}$$

For the nondiagonal mass matrix, we have

$$D = \left[\begin{array}{c|c} M^{-1}(M\tilde{\Omega} + \tilde{\Omega}M) & M^{-1}(K + K_G + \tilde{\Omega}M\tilde{\Omega}) \\ \hline -I & 0 \end{array} \right] \tag{20}$$

In the case of a diagonal mass representation, the matrix D reduces to

$$D = \left[\begin{array}{c|c} 2\tilde{\Omega} & M^{-1}(K + K_G) + \tilde{\Omega}\tilde{\Omega} \\ \hline -I & 0 \end{array} \right] \tag{21}$$

For modal analysis of a rotating structure employing Eq. (21), the mass matrix occurs only in the $M^{-1}(K + K_G)$ term. This is the normal dynamic matrix for a flexible structure and is very simply computed for the case of a diagonal mass matrix. Formation of the D matrix defined by Eq. (20) requires three times as many matrix multiplications as required by the Eq. (21) formulation.

Summary

The development of the governing equations of motion for a rotating flexible structure have been presented. Included in this development was a recognition of the terms resulting with the use of a nondiagonal mass matrix. As is pointed out, there is added computatinal complexity associated with the use of the nondiagonal mass formulation.

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Displacement Dependent Friction in Space Structural Joints

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Introduction

ONE of the less well understood aspects of the dynamics of structures is the origin and characteristic of their structural damping. At least three sources of passive energy dissipation exist in a space structure: material damping, passive damping elements,[1,2] and joints and fittings. Energy dissipation in joints takes place due to the relative motion of the contacting surfaces, which leads to both frictional and impact losses.

Ideally, the joints of a multielement structure are designed either to stiffen the ends of the structural members, as in a clamped end, or to allow the members one or more degrees of freedom, as in a pinned end. Realistically, motion occurs in the *rigid* joint since bending in the structural member causes the joint to deform elastically and motion is impeded in the *pinned* joint since friction occurs about the pin. In both cases, motion in the joint results in energy dissipation. This Note focuses on modeling the energy dissipation in two representative joint models. The model of friction used is one with a nonconstant coefficient. The frictional coefficient is zero at the mean displacement and varies linearly with the absolute magnitude of displacement.

Displacement Dependent Frictional Damping

One measure of damping useful in analyzing friction is the loss coefficient g related to the energy loss per cycle ΔE divided by the peak strain energy E

$$g=\frac{\Delta E}{2\pi E}=\frac{1}{2\pi E}\int F_f(\xi)\mathrm{d}\xi \tag{1}$$

where $F_f(\xi)$ is the frictional force. In the case of Coulomb friction, the friction force is constant and proportional to the normal load. The energy loss per cycle varies linearly with amplitude and the total energy of the system varies with the square of the amplitude. Therefore, the loss coefficient varies inversely with displacement. If the load normal to the frictional surfaces in joints is due to gravity or a mechanical preload, the conventional model of constant normal load may be appropriate.

In the absence of gravity, the origin of the normal load in a space structure joint may be due to local or global deformation of the structure.[3] If the normal force varies linearly with the absolute value of displacement, the dependence of the friction force on displacement is

$$F_f=-\mu c\,|\xi|\,\mathrm{sign}(\dot{\xi}) \tag{2}$$

where the force is assumed to be zero with no deflection. With this frictional model, the loss per cycle ΔE varies as the square of the amplitude of displacement. Thus, displacement dependent friction yields the same functional dependence on motion as the conventional hysteretic model of material damping. For this case, where the friction force varies linearly with displacement and there is no mean load on the joint, the loss coefficient is independent of amplitude, as in the case of the conventional model of material damping.

Beam/Sleeve Joint

One joint concept for use in an erectable space structure is the beam/sleeve joint. In this concept, a concentric outer sleeve is used to stiffen a beam in bending. Both the beam and sleeve are compliant and, as the beam deflects, a moment is exerted on the sleeve, locally deforming it. An idealized model of the beam/sleeve joint is shown in Fig. 1. The beam is assumed to be simply supported at $x=0$ and elastically restrained by the joint at $x=\ell$. As the beam deflects, the corner of the beam rotates and deforms the joint, which has an effective spring constant k_j. The energy dissipation occurs due to friction between the corner of the beam and the joint. For small displacements, the friction force is dependent on the displacement of the beam

$$F_f=-\mu k_j\ell_j\,|w'(\ell,t)|\,\mathrm{sign}[\dot{w}'(\ell,t)] \tag{3}$$

The boundary conditions for the assumed Bernoulli-Euler beam equations are those for a simply supported beam with an elastic restoring moment applied at the joint ($x=\ell$),

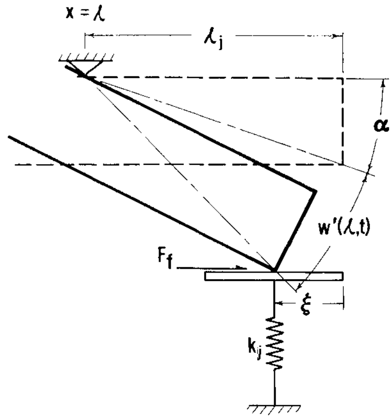

Fig. 1 Beam/sleeve joint model.

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